Recall: a second-order, linear, homogeneous differential equation is an equation of the form:

\[ P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x) y = 0. \]

If the functions \( P(x), Q(x), \) and \( R(x) \) are constant, we call the equation a "constant coefficient" second-order, linear, homogeneous differential equation. Let us assume the \( P, Q, \) and \( R \) are indeed constant, say \( P(x) = a, Q(x) = b \) and \( R(x) = c \). That is, the differential equation becomes

\[ a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \]

Solutions to higher–order, linear, homogeneous ODE’s are in the form of \( y = e^{mx} \). Substitute \( y = e^{mx} \) into the equation \( ay^{''} + by^{'} + cy = 0 \) and show that the equation reduces to the form of the corresponding auxiliary (or characteristic) equation given by \( am^2 + bm + c = 0 \).

For each of the following constant-coefficient, second-order, linear, homogeneous differential equations, write down the corresponding auxiliary differential equation. Then solve for \( m \). Write the general solution to the ODE.

1. \( y^{''} + 4y^{'} + 3y = 0 \)
2. \( y^{''} - 2\pi y^{'} - 8\pi^2 y = 0 \)

Repeated Solutions
Solve for \( m \). Use Reduction of Order to find a second linearly independent solution to the ODE.
3. \( y^{''} + 4y^{'} + 4y = 0 \)
If a linear, homogeneous ODE with constant coefficients has an auxiliary equation with a solution of \( m = \alpha \), multiplicity \( n \), then solutions to the ODE are: \( y_1 = e^{\alpha x} \), \( y_2 = xe^{\alpha x} \), \( y_3 = x^2 e^{\alpha x} \), \ldots , \( y_n = x^{n-1} e^{\alpha x} \).

Solve each ODE.
4. \( y'' - 6y' + 9y = 0 \)

5. \( y^{(4)} + 4y''' + 4y'' = 0 \)

**Imaginary Solutions.**
6. Recall that \( e^{ix} = \cos x + i \sin x \). Use this to explain why \( \cos x = \frac{1}{2} (e^{ix} + e^{-ix}) \) and \( \sin x = \frac{1}{2i} (e^{ix} - e^{-ix}) \).

If the solutions to the auxiliary equation are \( m = \alpha \pm \beta i \), then the general solution to the ODE will then be of the form \( y = e^{\alpha x} [c_1 \cos (\beta x) + c_2 \sin (\beta x)] \).

7. Find the solutions to \( y'' + 4y' + 6y = 0 \)
Nonhomogeneous, Linear, Higher-Order ODE’s with Constant Coefficients.

Suppose we have a non-homogeneous constant-coefficient differential equation

\[ a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = G(x). \]

Let \( y_c(x) \) be the complementary solution to \( a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \) and \( y_p(x) \) be the particular solution to

\[ a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = G(x). \]

Then the general solution is \( y(x) = y_c(x) + y_p(x) \).

8. Given \( G(x) \) we want to be able to guess a suitable \( y_p(x) \). Fill in the table below with guesses for \( y_p(x) \) (using the method of undetermined coefficients):

<table>
<thead>
<tr>
<th>If ( G(x) = )</th>
<th>guess for ( y_p(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^x )</td>
<td></td>
</tr>
<tr>
<td>( x )</td>
<td></td>
</tr>
<tr>
<td>( x^3 )</td>
<td></td>
</tr>
<tr>
<td>( x^3 e^{-2x} )</td>
<td></td>
</tr>
<tr>
<td>( \sin 3x )</td>
<td></td>
</tr>
<tr>
<td>( x^2 \sin x )</td>
<td></td>
</tr>
<tr>
<td>( x^3 + xe^{x} )</td>
<td></td>
</tr>
<tr>
<td>( e^{-x} \cos 2x )</td>
<td></td>
</tr>
<tr>
<td>( e^x \sin x - 3 e^x \cos x )</td>
<td></td>
</tr>
<tr>
<td>( 5 - 2x - 3e^{4x} + \cos 2x )</td>
<td></td>
</tr>
</tbody>
</table>

9. Solve. \( 5y'' + y' = -6e^{2x} \), \( y(0) = 0, \ y'(0) = -10 \)
Warning!!!
If your guess is already a solution to the complementary homogeneous problem, you must multiply by $x^r$ where $r$ is the smallest power so that no term in the particular solution is of the same form as a term in the complementary solution.

10. $y'' - y' - 2y = e^{-x}$

11. Solve. $y'' + 4y' + 4y = xe^{-2x} + \cos 2x$